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LATTICE POINTS AND LIE GROUPS. I

BY

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ABSTRACT. Assume that G is a compact semisimple Lie group and $\mathfrak{G}$ its associated Lie algebra. It is shown that the number of irreducible representations of G of dimension less than or equal to n is asymptotic to $kn^{a/b}$, where $a =$ the rank of $\mathfrak{G}$ and $b =$ the number of positive roots of $\mathfrak{G}$.

Let G be a simple, compact or complex, simply connected Lie group and $\mathfrak{G}$ its associated Lie algebra. If G is compact a representation is a real analytic group homomorphism $f: G \to GL(V)$ where V is a complex vector space. If G is complex a representation is a complex analytic group homomorphism $f: G \to GL(V)$. In either case f will be called irreducible if V has no nontrivial invariant subspaces under the action of $f(G)$. A homomorphism of Lie groups induces a homomorphism of the associated Lie algebras,

$$f^*: \mathfrak{G} \to \mathfrak{G}l(V),$$

a Lie algebra representation, and f^* will be called irreducible if V has no nontrivial invariant subspaces under the action of $f^*(\mathfrak{G})$. It is seen from this definition that f is irreducible $\leftrightarrow f^*$ is irreducible. If G is simply connected a Lie algebra representation of $\mathfrak{G}$ induces a group representation of G and we thus have a bijection between irreducible representations of G and $\mathfrak{G}$. By the dimension of a representation we mean the dimension of V. Identifying conjugate representations we ask, "How many irreducible representations of G (or equivalently $\mathfrak{G}$) are of dimension $\leq T$?". The question is simpler when asked of Lie algebras since the structure of the representations is less complex.

$\mathfrak{G}$ is a complex simple Lie algebra if G is a complex simple Lie group or a compact real form of a complex simple Lie algebra when G is a compact simple Lie group. In the latter case there is a bijection between the complex representations of $\mathfrak{G}$ defined over $\mathbf{R}$ and the complex representations of its complexifications, $\mathfrak{G} \otimes \mathbf{C}$, a complex simple Lie algebra so that we need only consider the case of $\mathfrak{G}$ complex and simple.

The root space decomposition of a simple complex Lie algebra is well known

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(1) The results in this paper constitute part of the author's thesis.

and is found in [1] and [2]. We let $\mathfrak{H}$ be a Cartan subalgebra, $\mathfrak{H}^*$ its dual and $\mathfrak{G} = \mathfrak{H} \oplus_\alpha \mathfrak{G}_\alpha$ be the canonical root space decomposition of $\mathfrak{G}$,

$$\mathfrak{G}_\alpha = \{X \in \mathfrak{G} \mid [H, X] = \alpha(H)X,\ H \in \mathfrak{H}\}.$$

$R = \{\alpha \in \mathfrak{H}^* \mid \mathfrak{G}_\alpha \neq 0\}$ is called the set of roots. A subset of R, $\{\alpha_1, \cdots, \alpha_{a_\mathfrak{G}}\}$, will be called simple if they are linearly independent, span $\mathfrak{H}^*$ and form an integer basis for R. The dimension of $\mathfrak{H} = a_\mathfrak{G}$ is the rank of $\mathfrak{G}$.

The Killing form is defined by $(X, Y) = \mathrm{Tr}(\mathrm{Ad}\,X \circ \mathrm{Ad}\,Y)$. Restricted to $\mathfrak{H}$ it is symmetric and nondegenerate. $(\ ,\ )$ induces a dual form on $\mathfrak{H}^*$ so we may speak of (α, β) when α and β are roots. Further, there are unique vectors H_α, $H_\beta \in \mathfrak{H}$ such that $(\alpha, \beta) = \alpha(H_\beta) = \beta(H_\alpha) = (H_\alpha, H_\beta)$.

If $f^*: \mathfrak{G} \to \mathfrak{G}l(V)$ is a representation it has a weight space decomposition, $V = \bigoplus_\lambda V_\lambda$, where

$$V_\lambda = \{v \neq 0 \mid f^*(H)v = \lambda(H)v, \text{ any } H \in \mathfrak{H}\}.$$

If f^* is finite dimensional it is necessary that

$$\lambda(H_i) = \lambda(2H_{\alpha_i}/(\alpha_i, \alpha_i)) = 2(\lambda, \alpha_i)/(\alpha_i, \alpha_i) \in \mathbf{Z}$$

for any α_i, $i = 1, \cdots, a$. If f^* is irreducible there exists a weight λ, called the dominant weight, such that $\lambda \geq \lambda'$ for any other λ' in f^* and $\lambda(H_i) \in \mathbf{Z}^+$, $i = 1, \cdots, a$. Furthermore, if $f^{*\prime}$ is another irreducible representation with λ as dominant weight then f^* is conjugate to $f^{*\prime}$. Thus we may identify f^* with its dominant weight and we will write π_λ for f^*. The lattice of dominant weights is $\mathbf{Z}^+\lambda_1 \oplus \cdots \oplus \mathbf{Z}^+\lambda_a$ where $\lambda_i(H_j) = \delta_{ij}$. The interest of this is that the dimension of π_λ is a polynomial in λ. By the Weyl character formula

$$f'_\mathfrak{G}(\lambda) = \dim \pi_\lambda = \prod_{\alpha>0} (\lambda + \delta, \alpha) \Big/ \prod_{\alpha>0} (\delta, \alpha)$$

where $\delta = \tfrac{1}{2}\Sigma_{\alpha>0}\alpha \cdot \delta = \Sigma\lambda_i$ [1, p. 257], so if λ belongs to the lattice of dominant weights then $\lambda + \delta$ belongs to the lattice of dominant weights. If we change coordinates to $\Lambda = \lambda + \delta = \Sigma\, \Lambda_i \lambda_i$ where $\Lambda_i \in \mathbf{R}$, then

$$\dim \pi_\lambda = f_\mathfrak{G}(\Lambda) = \prod_{\alpha>0} (\Lambda, \alpha) \Big/ \prod_{\alpha>0} (\delta, \alpha).$$

The number of irreducible representations of $\mathfrak{G}$ of dimension $\leq n$ is then equal to the number of lattice points, Λ, such that $\Lambda_i > 0$ and $f_\mathfrak{G}(\Lambda) \leq n$. We now state

Theorem. *Let G be a simply connected, simple, complex or compact Lie group. The number of irreducible representations of G of dimension $\leq n$ is asymptotic to $kn^{a_\mathfrak{G}/b_\mathfrak{G}}$; $b_\mathfrak{G}$ = the number of positive roots of $\mathfrak{G}$.*

Proof. We first note that (Λ, α) is a linear homogeneous polynomial in the coefficients of Λ since

$$\left(\sum_{i=1}^{a} X_i\lambda_i,\ \sum_{i=1}^{a} m_i\alpha_i\right) = \sum_{i=1}^{a} m_i(\lambda_i,\ \alpha_i)X_i.$$

If $e_i, \cdots, e_a$ is an orthonormal basis of $\mathfrak{H}^*$ and if $M: \lambda_i \to e_i$, then if M^t is the transpose of M with respect to $(\ ,\ )$

$$(\Lambda,\ \alpha) = (M^{-1}M\Lambda,\ \alpha) = (M\Lambda,\ (M^{-1})^t\alpha)$$

and $M\Lambda$ lies in the regular integer lattice in $\mathbf{R}^a$. Thus if $L = \sum_{i=1}^a X_i e_i$, $X_i > 0$, and

$$f^0_{\mathfrak{G}}(L) = \prod_{\alpha>0}(L,\ (M^{-1})^t\alpha)\Big/\prod_{\alpha>0}(M\delta,\ M^{-1})^t\alpha)$$

then $f^0_{\mathfrak{G}}(\sum_{i=1}^a X_i e_i) = f_{\mathfrak{G}}(\sum_{i=1}^a X_i\lambda_i)$ so we may regard $f_{\mathfrak{G}}$ as having asymptotes $e_i = 0$ and the lattice of weights as the ordinary integer lattice. We now prove a lemma on homogeneous functions.

Lemma 1. *Let f be a homogeneous function on $\mathbf{R}^a$ of degree b which is the product of linear forms $\sum m_i x_i$, $m_i \geq 0$. If $f = 0$ on the planes $x_i = 0$, $i = 1, \cdots, a$, and if*

$$S(1) = \{x \in \mathbf{R}^a \mid f(x) \leq 1,\ x_i \geq 0\}$$

has finite volume then the number of lattice points in

$$S(r) = \{x \in \mathbf{R}^a \mid f(x) \leq r,\ x_i \geq 0\}$$

is asymptotic to $\operatorname{Vol}(S(1))r^{a/b}$.

Proof. It is clear that the volume of $S(r) = \operatorname{Vol}(S(1))r^{a/b}$. If $x \in S(r)$ then

$$f(x/(r^{1/b})) = (r^{-1/b})^b f(x) = r^{-1}f(x) \leq 1.$$

Since we are in $\mathbf{R}^a$ the Jacobian of the coordinate change $x \to \alpha x$ is α^a so $\operatorname{Vol}(S(r)) = r^{a/b}\operatorname{Vol}(S(1))$. We will be done if the number of lattice points in $S(r) \sim \operatorname{Vol}(S(r))$. To see this, draw a unit a-cube at every lattice point of $S(r)$, w, with vertices at w, $w + e_i$ any i. Call the union of these cubes $\overline{L}(r)$; a set which will contain $S(r) \cap \{x_i \geq 1 \text{ all } i\}$ since f will be increasing in each coordinate. Now at each lattice point, w, draw a unit cube with vertices w, $w - e_i$ any i. Call the union of these cubes $\underline{L}(r)$. $\underline{L}(r) \subset S(r)$ and $\operatorname{Vol}\underline{L}(r) = \operatorname{Vol}\overline{L}(r)$. Call $E(r) = S(r) \cap \{x_i \leq 1 \text{ some } i\}$. Then

$$\underline{L}(r) \subset S(r) \subset \overline{L}(r) \cup E(r)$$

which implies $|\operatorname{Vol} S(r) - \text{the number of lattice points}| \leq \operatorname{Vol} E(r)$. However

$$\operatorname{Vol} E(r) = r^{a/b}\operatorname{Vol}\{x \in S(1) \mid x_i \leq r^{-1/b} \text{ some } i\}$$

and since Vol $S(1) < \infty$ the volume of this latter set $\to 0$ by dominated convergence. Thus Vol $E(r)$ is $o(\mathrm{Vol}\, S(r))$ and the number of lattice points in $S(r)$ is asymptotic to Vol $S(r)$. □

We now have a criterion we would like to apply to the polynomials $f_{\mathfrak{G}}$. A canonical example is the algebra A_2. The positive roots of A_2 are $\alpha_1, \alpha_2, \alpha_1 + \alpha_2$ and the polynomial $f^0_{A_2}(x, y) = kxy(x+y)$. We wish to show

$$\mathrm{Vol}\{x, y \mid x > 0, y > 0, kxy(x+y) \leq 1\} < \infty$$

or equivalently Vol $A < \infty$ where

$$A = \{x, y \mid x > 0, y > 0, xy(x+y) \leq 1\}.$$

We divide A into two subsets, $A_x = A \cap \{x \geq y\}$, $A_y = A \cap \{x \leq y\}$. If $(x, y) \in A_x$, $xy(x+y) \leq 1$ which implies $x^2 y \leq 1$.

$$A_x \subset \{(x, y) \mid x > y > 0, x^2 y \leq 1\}.$$

Vol $A_x \cap \{x \in [0, 1]\} \leq \frac{1}{2}$ so Vol A_x is finite if

$$\mathrm{Vol}\{(x, y) \mid x > y, x > 1, x^2 y \leq 1\} < \infty.$$

The volume of this set is $\int_1^\infty x^{-2}\, dx = 1$ so Vol $A_x \leq 3/2$. Similarly, Vol $A_y \leq 3/2$ so Vol $A < 3$ and the theorem is true for the algebra A_2. We now extend this method to higher dimensions.

Lemma 2. *In* $\mathbf{R}^a$ *let* $f(x)$ *be a sum of monomials of degree* b. *If for every permutation* i *of* $\{1, \cdots, a\}$ *there exists in* $f(x)$ *a monomial* $X^{s_1}_{i(1)} \cdots X^{s_a}_{i(a)}$ *where* $s_1 > \cdots > s_a > 0$, *then the volume of the set* $S(1) = \{x \mid f(x) \leq 1, x_i \geq 0\}$ *is finite.*

Remark. From Lemma 1 this implies Vol $S(r) = \mathrm{Vol}\, S(1) r^{a/b}$.

Proof of Lemma 2. We proceed by induction. If $a = 2$ we have monomials $X_1^{s_1} X_2^{s_2}$ and $X_1^{s'_2} X_2^{s'_1}$, $s_1 > s_2$, $s'_1 > s'_2$. Again partitioning $S(1)$ into A_x and A_y we see

$$\mathrm{Vol}\, A_x \leq \tfrac{1}{2} + \int_1^\infty x^{-s_1/s_2}$$

$$= \tfrac{1}{2} + (s_1/s_2 - 1)^{-1} < \infty \quad \text{since } s_1 > s_2.$$

Similarly Vol $A_y \leq \frac{1}{2} + (s'_1/s'_2 - 1)^{-1}$.

Now assume the lemma true for $a - 1$. Partition $S(1)$ into the sets

$$A_{i_1, \ldots, i_a} = S(1) \cap \{x_{i_1} \geq \cdots \geq x_{i_a}\}.$$

We wish to show Vol $A_{i_1, \cdots, i_a} < \infty$ for any i. As before

$$A_{i_1,\ldots,i_a} \subset \{x \mid x_{i_1} \geq \cdots \geq x_{i_a},\ x_{i_1}^{s_1} \cdots x_{i_a}^{s_a} \leq 1\}.$$

If $x_{i_1} \geq 1$ a cross-section of this set at x_{i_1} is the set

$$\{(x_{i_2}, \cdots, x_{i_a}) \mid x_{i_2} \geq \cdots \geq x_{i_a} \geq 0,\ x_{i_2}^{s_2} \cdots x_{i_a}^{s_a} \leq 1/x_{i_1}^{s_1}\}.$$

By induction and the previous remark the volume of the cross-section $= kx_{i_1}^{-\gamma}$ where $\gamma = s_1(a-1)/(\Sigma_{i=2}^a s_i)$. The volume of

$$A_{i_1,\ldots,i_a} \leq \mathrm{Vol}(A_{i_1,\ldots,i_a} \cap \{x_{i_1} \in [0, 1]\}) + \int_1^\infty y^{-\gamma}\, dy.$$

The first set is contained in the unit cube so it has volume ≤ 1 and the integral is finite as long as $\gamma \geq 1$. But $s_1 > s_i \forall i > 1$ so $(a-1)s_1 > \Sigma_{j=2}^a s_j \Rightarrow \gamma > 1$. □

The proof of Theorem 1 will be complete if we show the criterion of Lemma 2 applies to the polynomials $f_{\mathfrak{G}}$ for all simple complex Lie algebras.

If $\Lambda = \Sigma_{i=1}^a X_i \lambda_i$ then for each $\alpha = \Sigma_{i=1}^a m_i \alpha_i$

$$(\Lambda, \alpha) = \sum_{i=1}^a m_i (\lambda_i, \alpha_i) X_i.$$

Thus to determine f we must list all the positive roots of $\mathfrak{G}$ in terms of the simple roots. We begin with the A_n algebras.

Lemma 3. *The monomial* $X_1^{s(1)} \cdots X_n^{s(n)}$ *is found in the expansion of* f_{A_n} *for every permutation* s *of* $(1, \cdots, n)$.

Proof. By referring to Serre [2] the positive roots of A_n are $\alpha_1, \cdots, \alpha_n$; $\alpha_1 + \alpha_2, \cdots, \alpha_{n-1} + \alpha_n; \cdots; \alpha_1 + \cdots + \alpha_n$. Since $(\lambda_i, \alpha_i) = c$, $f_{A_n} = kX_1 \cdots X_n(X_1 + X_2) \cdots (X_{n-1} + X_n) \cdots (X_1 + \cdots + X_n)$. We now apply induction. If $n = 2$, $f_{A_2} = X_1^2 X_2 + X_1 X_2^2$. Now assume the lemma for $n-1$. We write $f_{A_n} = X_n(X_n + X_{n-1}) \cdots (X_1 + \cdots + X_n) f_{A_{n-1}}$. Pick an arbitrary permutation s. Then $s(n) = j$. By induction $X_1^{s(1)'} \cdots X_{n-1}^{s(n-1)'}$ occurs in $f_{A_{n-1}}$ where

$$s(i)' = \begin{cases} s(i) & \text{if } s(i) < j, \\ s(i) - 1 & \text{if } s(i) > j. \end{cases}$$

Multiply this monomial by X_n in the first j factors $X_n, \cdots, (X_n + \cdots + X_{n+j-1})$. Now pick the least i such that $s(i)' < s(i)$. Multiply the monomial by X_i in $(X_1 + \cdots + X_n)$. Then pick the next i' such that $s(i') < s(i')'$ and multiply by $X_{i'}$ in $(X_2 + \cdots + X_n)$. Since $i' > i \Rightarrow i' \geq 2$, $X_{i'}$ is found in $(X_2 + \cdots + X_n)$. We may thus continue until we have $X_1^{s(1)} \cdots X_n^{s(n)}$. □

Remark. The degree of f_{A_n} is minimal such that we may find monomials

$X_{i(1)}^{s1} \cdots X_{i(n)}^{sn}$ where $s_1 > \cdots > s_n > 0$ since $s_n \geq 1,\ s_{n-1} \geq 2,\ \cdots,\ s_1 \geq n$ so that the degree of $f = \Sigma_{i=1}^{n} s_i \geq \Sigma_{i=1}^{n} i =$ the degree of f_{A_n}

Lemma 4. *The monomial* $X_1^{2s(1)-1} \cdots X_n^{2s(n)-1}$ *is found in the polynomials* f_{B_n} *and* f_{C_n} *for any permutation s.*

Proof. The positive roots of B_n are $\alpha_1, \cdots, \alpha_n; \alpha_1 + \alpha_2, \cdots, \alpha_{n-1} + \alpha_n; \cdots; \alpha_1 + \cdots + \alpha_n$ and $\alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_n$ where $i < j \leq n$ [2]. $f_{B_n} = kf_{A_n} \Pi_{i=1}^{n-1} \Pi_{j=i+1}^{n} (X_i + \cdots + X_{j-1} + 2X_j + \cdots + 2X_n)$. From Lemma 3 we know the monomial $X_1^{s(1)} \cdots X_n^{s(n)}$ is in f_{A_n}. We wish then to show that $X_1^{s(1)-1} \cdots \hat{X}_j \cdots X_n^{s(n)-1}$ where $s(j) = 1$ lies in

$$\prod_{i=1}^{n-1} \prod_{j=j+1}^{n} (X_i + \cdots + X_{j-1} + 2X_j + \cdots + 2X_n).$$

We proceed as follows. There are $n-1$ factors containing X_1. $s(1) - 1 \leq n - 1$ so we may choose X_1 in $s(1) - 1$ of these factors. There are $(n-1) + (n-2)$ factors containing X_2 and

$$(s(1) - 1) + (s(2) - 1) \leq (n-1) + (n-2)$$

so choose X_2 in the next $s(2) - 1$ factors. Thus we may proceed at each stage being able to choose $s(i) - 1$ X_i's. Multiplying we have the monomial $X_1^{2s(1)-1} X_2^{2s(2)-1} \cdots X_n^{2s(n)-1}$.

For C_n the positive roots are $\alpha_1, \cdots, \alpha_n; \alpha_1 + \alpha_2, \cdots, \alpha_{n-1} + \alpha_n; \cdots; \alpha_1 + \cdots + \alpha_n; \alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{n-1} + \alpha_n$, $i < n$, $i \leq j \leq n-1$. The roots are different from B_n but contain the same α_i so the argument is the same. □

Lemma 5. f_{D_n} *contains monomials of descending degrees for* $n \geq 6$.

Proof. Referring to Serre the positive roots of D_n are $\alpha_1, \cdots, \alpha_{n-1}; \alpha_1 + \alpha_2, \cdots, \alpha_{n-2} + \alpha_{n-1}; \cdots; \alpha_1 + \cdots + \alpha_{n-1}; \alpha_{n-2} + \alpha_n, \cdots, \alpha_1 + \cdots + \alpha_{n-2} + \alpha_n; \alpha_{n-2} + \alpha_{n-1} + \alpha_n, \cdots, \alpha_1 + \cdots + \alpha_{n-1} + \alpha_n; \alpha_i + \cdots + 2\alpha_j + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$. We may write

$$f_{D_n} = kf_{A_{n-1}}(X_{n-2} + X_n)(X_{n-3} + X_{n-2} + X_n) \cdots (X_1 + \cdots + X_{n-2} + X_n)$$

$$\cdot \underbrace{X_n(X_{n-2} + X_{n-1} + X_n) \cdots (X_1 + \cdots + X_n)}$$

$$\cdot \prod_{i=1}^{n-3} \prod_{j=i+1}^{n-2} (X_i + \cdots + 2X_j + \cdots + 2X_{n-2} + X_{n-1} + X_n).$$

The bracketed expression is what is needed along with $f_{A_{n-1}}$ to create f_{A_n} except for the missing factor $(X_{n-1}+X_n)$. We compensate by adding the term $(X_{n-3}+2X_{n-2}+X_{n-1}+X_n)$ to create a function containing every monomial of f_{A_n}. The remaining terms we write as

$$g_{D_n}=\prod_{i=1}^{n-2}(X_i+\cdots+X_{n-2}+X_n)$$

$$\prod_{i=1}^{n-4}\prod_{j=i+1}^{n-2}(X_i+\cdots+2X_j+\cdots+2X_{n-2}+X_{n-1}+X_n).$$

We know from Lemma 3 that $X_{s(1)}^1\cdots X_{s(n)}^n$ is found in f_{A_n} for any permutation s. We wish to produce a monomial with descending degrees in the $X_{s(i)}$ in g_{D_n} for any permutation s. There are two cases. First assume that $s(1)\neq n-1$. Then we will be done if the monomial

$$X_{s(n)}^{n-2}\cdots X_{s(6)}^4X_{s(5)}^2X_{s(4)}X_{s(3)}X_{s(2)}$$

is in g_{D_n}. First choose $n-2$ different X_i from

$$\prod_{i=1}^{n-2}(X_i+\cdots+X_{n-2}+X_n),\qquad i\neq n-1,\ s(1).$$

We then proceed to the second factor. There are $n-3$ terms containing X_1 so if $s(j)=1$ we may pick X_1 in $j-3$ terms. Mimicking Lemma 4 we may continue by picking $j'-3$ X_2's; where $s(j')=2$ and so on to X_{n-2}. The sole difference in the procedure will be that if $j\in(1, 2, 3, 4)$ we choose no $X_{s(j)}$'s. After X_{n-2} every term contains X_{n-1} and X_n so we may arbitrarily choose $k-2$ X_{n-1}'s and $k'-3$ X_n's; $s(k)=n-1$, $s(k')=n$. We have thus produced the desired monomial belonging to g_{D_n} and multiplying by $X_{s(1)}^1\cdots X_{s(n)}^n$ we have a monomial with strictly decreasing degrees.

If $n-1=s(1)$ we will be done if

$$X_{s(n)}^{n-3}X_{s(n-1)}^{n-3}\cdots X_{s(6)}^4X_{s(5)}^2X_{s(4)}X_{s(3)}X_{s(2)}X_{s(1)}$$

is in g_{D_n}. First pick $\{X_{s(n-1)},\cdots,X_{s(2)}\}$ in $\Pi_{i=1}^{n-2}(X_i+\cdots+X_{n-2}+X_n)$. Then proceed as before choosing $j-3$ X_1's, $j'-3$ X_2's and so on again skipping $X_{s(1)},\cdots,X_{s(4)}$. Proceed to X_{n-2} and then to X_n. There will be one remaining term which *a priori* contains X_{n-1}. Multiplying by X_{n-1} from this factor we produce our monomial.

We have proved Theorem 1 for A_n, B_n, C_n and D_n for $n\geq 6$. These are all the complex simple Lie algebras except for the algebras G_2, F_4, D_4, D_5, E_6, E_7 and E_8. In these cases the conditions of Lemma 2 may be verified directly.

We now summarize the results:

Algebra	$a_{\mathfrak{G}}$	$b_{\mathfrak{G}}$	$c_{\mathfrak{G}} = a_{\mathfrak{G}}/b_{\mathfrak{G}}$
$A_n, n \geq 1$	n	$n(n+1)/2$	$2/n+1$
$B_n, C_n, n \geq 2$	n	n^2	$1/n$
$D_n, n \geq 4$	n	$n(n-1)$	$1/n-1$
G_2	2	6	1/3
F_4	4	24	1/6
E_6	6	36	1/6
E_7	7	63	1/9
E_8	8	120	1/15

We now extend our results to semisimple Lie algebras.

Corollary. *Let $\mathfrak{G}$ be a semisimple Lie algebra, $\mathfrak{G} = \bigoplus_{i=1}^n \mathfrak{G}_i$, with $\mathfrak{G}_i$ the simple components. If $c_{\mathfrak{G}_1} = \cdots = c_{\mathfrak{G}_s} > c_{\mathfrak{G}_{s+1}} \geq \cdots \geq c_{\mathfrak{G}_n}$, then the number of irreducible representations of $\mathfrak{G}$ of dimension less than or equal to T is asymptotic to $kT^{c_{\mathfrak{G}_1}} \log^{s-1} T$.*

Proof. We first assume that $\mathfrak{G}$ has two simple factors, $\mathfrak{G} = \mathfrak{G}_1 \oplus \mathfrak{G}_2$. The irreducible representations of $\mathfrak{G}$ are tensor products of irreducible representations of the simple factors and the dimension of the tensor representation is a product of the dimensions of the factor representations. The number of irreducible representations of $\mathfrak{G}$ of dimension $\leq r$ is $h(r) = \Sigma_{m,n \in Z^+; mn \leq r} M_1(m) M_2(n)$, where $M_i(x)$ is the number of irreducible representations of $\mathfrak{G}_i$ of dimension x.

We partition $S = \{x, y \mid xy \leq r,\ x, y \geq 0\}$ into $S_x = S \cap \{x \in [0, r^{1/2}]\}$, $S_y = S \cap \{y \in [0, r^{1/2}]\}$. $S = S_x \cup S_y$ so if we estimate both $h_x(r) = \Sigma_{(m,n)\in S_x} M_1(m) M_2(n)$ and $h_y(r) = \Sigma_{(m,n)\in S_y} M_1(m) M_2(y)$ asymptotically, then $h(r) \sim \mathrm{Max}(h_x(r), h_y(r))$. Assume $c_{\mathfrak{G}_1} > c_{\mathfrak{G}_2}$ (c_1 and c_2 for brevity); we will deal with $c_1 = c_2$ later. Theorem 1 states $\Sigma_{j=1}^n M_i(j) \sim \mu_i n^{c_i}$. Thus

$$h_x(r) = \sum_{i=1}^{[r^{1/2}]} M_1(i) \sum_{j=1}^{[r/i]} M_2(j).$$

For any ϵ there exists r_2 such that

$$\left| \left(\sum_{j=1}^{L} M_2(j) - \mu_2 L^{c_2} \right) \Big/ \sum_{j=1}^{L} M_2(j) \right| < \epsilon \quad \text{any } L \geq r_2.$$

Then

$$h_x(r) = \mu_2 r^{c_2} \sum_{i=1}^{[r^{1/2}]} M_1(i)/i^{c_2} + \epsilon' h_x(r)$$

where $|\epsilon'| < \epsilon$ if $r > r_2^2$. Thus

$$h_x(r) \sim \mu_2 r^{c_2} \sum_{i=1}^{[r^{1/2}]} M_1(i)/i^{c_2}.$$

By the Abel summation formula

$$\sum_{i=1}^{[r^{1/2}]} M_1(i)/i^{c_2} = \sum_{i=1}^{[r^{1/2}]-1} \left(\sum_{j=1}^{i} M_1(j) \right) (1/i^{c_2} - 1/(i+1)^{c_2}) + \sum_{i=1}^{[r^{1/2}]} M_1(i) \cdot r^{-c_2/2}.$$

Now $c_2/i^{c_2+1} > 1/i^{c_2} - 1/(i+1)^{c_2} > c_2/(i+1)^{c_2+1}$, so

$$\sum_{i=1}^{[r^{1/2}]-1} \left(\sum_{j=1}^{i} M_1(j) \right) \cdot c_2/i^{c_2+1}$$

$$> \sum_{i=1}^{[r^{1/2}]-1} \left(\sum_{j=1}^{i} M_1(j) \right) (1/i^{c_2} - 1/(i+1)^{c_2}) > \sum_{i=1}^{[r^{1/2}]-1} \left(\sum_{j=1}^{i} M_1(j) \right) \cdot c_2/(i+1)^{c_2+1}.$$

For any $\epsilon > 0$ there exists r_1 such that

$$\left| \left(\sum_{j=1}^{L} M_1(j) - \mu_1 L^{c_1} \right) \Big/ \sum_{j=1}^{L} M_1(j) \right| < \epsilon \quad \text{any } L \geq r_1.$$

If $r \gg r_1^2, r_2^2$

$$\sum_{i=1}^{[r^{1/2}]-1} \left(\sum_{j=1}^{i} M_1(j) \right) \cdot c_2/i^{c_2+1} = \sum_{i=1}^{[r^{1/2}]-1} \mu_1 c_2 i^{c_1-c_2-1} + E + A$$

where

$$|E| < \epsilon \sum_{i=r_1}^{[r^{1/2}]-1} \left(\sum_{j=1}^{i} M_1(j) \right) \cdot c_2/i^{c_2+1}$$

and

$$A = \sum_{i=1}^{r_1} \left(\sum_{j=1}^{i} M_1(j) - \mu_1 i^{c_1} \right) \cdot c_2/i^{c_2+1}.$$

$$\sum_{i=1}^{[r^{1/2}]-1} \mu_1 c_2 i^{c_1-c_2-1} \sim \mu_1 c_2 \int_1^{[r^{1/2}]-1} x^{c_1-c_2-1}\, dx$$

$$= \mu_1 c_2/(c_1 - c_2) x^{c_1-c_2} \Big|_1^{r^{1/2}-1} = k r^{(c_1-c_2)/2} + k'.$$

Also

$$\sum_{i=1}^{[r^{1/2}]} M_1(i) \cdot r^{-c_2/2} = k_0 r^{(c_1-c_2)/2} + E'$$

where $|E'| < \epsilon \sum_{i=1}^{[r^{1/2}]} M_1(i) \cdot r^{-c_2/2}$. Thus

$$\sum_{i=1}^{[r^{1/2}]} M_1(i)/i^{c_2} = (k + k_0)r^{c_1-c_2/2} + (k' + A) + (E + E').$$

From this

$$(1 + 2\epsilon)h_x(r) > (k + k')r^{c_1+c_2/2} + (k' + A)r^{c_2} > (1 - 2\epsilon)h_x(r).$$

Thus $h_x(r) \sim cr^{c_2} + c'r^{c_1+c_2}$. Similarly $h_y(r) \sim \mu_1 r^{c_1} \sum_{i=1}^{[r^{1/2}]} M_2(i)/i^{c_1}$. But in this case $\sum_{i=1}^{[r^{1/2}]} M_2(i)/i^{c_1}$ is asymptotic to a constant. To see this

$$\sum_{i=1}^{[r^{1/2}]} M_2(i)/i^{c_1} = \sum_{i=1}^{[r^{1/2}]-1} \left(\sum_{j=1}^{i} M_2(j) \right) (1/i^{c_1} - 1/(i+1)^{c_1}) + \sum_{j=1}^{[r^{1/2}]} M_2(j) r^{-c_1/2}.$$

$\sum_{j=1}^{x} M_2(j)$ is $O(x^{c_2})$ and $(1/i^{c_1} - 1/(i+1)^{c_1}) < c_1/i^{c_1+1}$, so

$$\sum_{i=1}^{[r^{1/2}]} M_2(i)/i^{c_2} \leq k \int_1^{r^{1/2}} x^{c_2-c_1-1}\, dx + k_0 r^{c_2-c_1/2}$$

$$= k/(c_1 - c_2)(1 - r^{c_2-c_1/2}) + k_0 r^{c_2-c_1/2}.$$

But $c_2 - c_1 < 0$ so the above sum is $\leq 2k/(c_1 - c_2)$ if r is sufficiently large and $\mathrm{Lim}_{r\to\infty} \sum_{i=1}^{r} M_2(i)/i^{c_1}$ exists and is equal to k'. Thus $h_y(r) \sim k' r^{c_1}$ and $h(r) \sim h_y(r)$.

This settles the case of $\mathfrak{G} = \bigoplus_{i=1}^{n} \mathfrak{G}_i$ where $c_1 > c_i$, $i > 1$. By the above argument $\mathfrak{G}_1 \oplus \mathfrak{G}_2$ has asymptotically $k' n^{c_1}$ irreducible representations of dimension $\leq n$. By iteration $(\mathfrak{G}_1 \oplus \mathfrak{G}_2) \oplus \mathfrak{G}_3$ still has $\sim k'' n^{c_1}$ irreducible representations and so on. This leaves the case of $c_1 = \cdots = c_s$. Let $\mathfrak{G} = \mathfrak{G}_1 \oplus \mathfrak{G}_2$. Tracing the argument for $c_1 \neq c_2$ nothing is changed until we arrive at $\int_1^{[r^{1/2}]} x^{c_1-c_2-1}\, dx$. This integral now equals $\int_1^{[r^{1/2}]} x^{-1}\, dx = \frac{1}{2} \log r$ so that $h_x(r) \sim kr^{c_1} \log r$ and $h_y(r) \sim k' r^{c_1} \log r$. Now

$$h(r) = h_x(r) + h_y(r) - \sum_{i,j \in S_x \cap S_y} M_1(i) M_2(j)$$

and the latter sum equals

$$\sum_{i=1}^{[r^{1/2}]} M_1(i) M_2(j) = \sum_{i=1}^{[r^{1/2}]} M_1(i) \cdot \sum_{j=1}^{[r^{1/2}]} M_2(j)$$

which is $O(r^{c_1})$ so that $h(r) \sim kr^{c_1} \log r$. Taking $\mathfrak{G} = (\mathfrak{G}_1 \oplus \mathfrak{G}_2) \oplus \mathfrak{G}_3$ we arrive at the integral

$$\int_1^{[r^{1/2}]} (\log x)/x \, dx = \frac{1}{8} \log^2 r.$$

So $h_x(r) \sim kr^{c_1} \log^2 r$, $h_y(r) \sim k' r^{c_1} \log^2 r$, $\sum_{i,j \in S_x \cap S_y} M_1(i) M_2(j)$ is $O(r^{c_1} \log r)$ and $h(r) \sim k_0 r^{c_1} \log^2 r$. Continuing to the case $\mathfrak{G} = \bigoplus_{i=1}^s \mathfrak{G}_i$ we have $h(r) \sim kr^{c_1} \log^{s-1} r$ and our corollary is proven. □

BIBLIOGRAPHY

1. N. Jacobson, *Lie algebras*, Interscience Tracts in Pure and Appl. Math., no. 10, Interscience, New York, 1962. MR 31 #2354.

2. J.-P. Serre, *Algèbres de Lie semi-simples complexes*, Benjamin, New York, 1966. MR 35 #6721.

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